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ERROR ESTIMATES IN L^2, H^1 AND L^∞ IN COVOLUME METHODS FOR ELLIPTIC AND PARABOLIC PROBLEMS: A UNIFIED APPROACH

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ABSTRACT. In this paper we consider covolume or finite volume element methods for variable coefficient elliptic and parabolic problems on convex smooth domains in the plane. We introduce a general approach for connecting these methods with finite element method analysis. This unified approach is used to prove known convergence results in the H^1, L^2 norms and new results in the max-norm. For the elliptic problems we demonstrate that the error $u - u_h$ between the exact solution u and the approximate solution u_h in the maximum norm is $O(h^2|\ln h|)$ in the linear element case. Furthermore, the maximum norm error in the gradient is shown to be of first order. Similar results hold for the parabolic problems.

1. INTRODUCTION

Let Ω be a convex domain in R^2 with smooth boundary $\partial\Omega$ and consider the general self-adjoint second order elliptic problem

$$Lu := -\sum_{i,j}^{2} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) + qu = f, \qquad x \in \Omega, \tag{1.1}$$

$$u = 0, \qquad x \in \partial\Omega, \tag{1.2}$$

where $q \in L^\infty$ is nonnegative, $f \in L^2(\Omega)$, and the matrix of coefficients $A := (a_{ij})$, $a_{ij} = a_{ji} \in W^{1,\infty}(\Omega)$ is uniformly elliptic; i.e., there exists a positive constant $r > 0$ such that

$$\sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \geq r(\xi_1^2 + \xi_2^2) \qquad \forall \xi := (\xi_1, \xi_2) \in R^2 \qquad a.e. \text{ in } \Omega. \tag{1.3}$$

The natural variational problem associated with (1.1)-(1.2) is to find $u \in U := H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \quad \forall v \in U, \tag{1.4}$$

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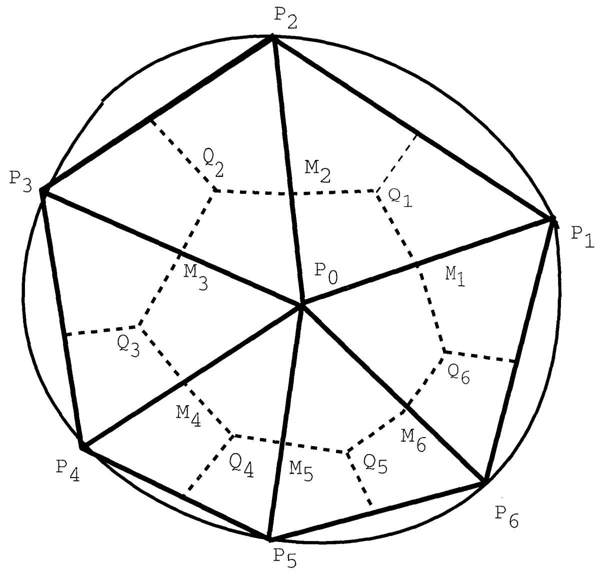

FIGURE 1. Primal and dual partitions of a convex domain

where

$$a(u,v) := \int_\Omega (\sum_{i,j}^{2} a_{ij} \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + quv)dx, \tag{1.5}$$

$$(f,v) = \int_\Omega fvdx. \tag{1.6}$$

Since the error estimates to be derived below require that the exact solution u be in $H^2(\Omega)$ for the H^1 norm case and be in $H^3(\Omega)$ for the max-norm and L^2 norm cases, it is necessary to have the smooth boundary assumption on the problem domain. If instead we were to consider a polygonal problem domain, all interior angles of the domain would have to be no greater than $\frac{\pi}{2.5}$ even if $f \in C^\infty$, rendering the L^2 and max-norm estimates so obtained too limited to be useful.

Referring to Figure 1, let $\mathcal{T}_h = \cup K_Q$ be a triangulation of the polygonal domain $\Omega_h \subset \Omega$ into a union of triangular elements, where K_Q stands for the triangle whose barycenter is Q. Here $h := \max h_K$ is the maximum of the diameters h_K over all triangles. The nodes of a triangular element are its vertices. We further require that the vertices which lie on $\partial\Omega_h$ also lie on $\partial\Omega$, so that there exists a constant C independent of h satisfying

$$\operatorname{dist}(x, \partial\Omega) \le Ch^2 \qquad \forall x \in \Omega\backslash\Omega_h. \tag{1.7}$$

Associated with the primal partition $\mathcal{T}_h$, we define its dual partition $\mathcal{T}_h^*$ of Ω_h as follows. Let P_0 be an interior node and $P_i, i = 1, \dots, 6$ be its adjacent nodes, and $M_i := M_{0i}$ be the midpoint of $\overline{P_0P_i}$. Connect successively the points $M_1, Q_1, M_2, Q_2, \dots, M_6, Q_6, M_1$ to obtain the dual polygonal element $K_{P_0}^*$. Its nodes are defined to be $Q_i, i = 1, \dots, 6$. The dual element $K_{P_2}^*$ based at a typical boundary node P_2 is $M_{12}Q_1M_2Q_2M_{23}P_2$. Let $\bar{\Omega}_h$ denote the set of all nodes of $\mathcal{T}_h$; $\Omega_h^\circ := \bar{\Omega}_h - \partial\Omega$ the set of all interior nodes in $\mathcal{T}_h$, and S_Q and $S_{P_0}^*$ denote the areas of triangle K_Q and polygon $K^*_{P_0}$, respectively. Throughout this paper we shall assume the partitions to be quasi-uniform. There exist two positive constants C_1, C_2 independent of h such that

$$C_1 h^2 \leq S_Q \leq C_2 h^2, \qquad Q \in \Omega_h^*, \tag{1.8}$$

$$C_1 h^2 \leq S^*_{P_0} \leq C_2 h^2, \qquad P_0 \in \bar{\Omega}_h. \tag{1.9}$$

Corresponding to $\mathcal{T}_h$, we define the trial function space $U_h \subset H_0^1(\Omega)$ as the space of continuous functions on the closure of Ω which vanish outside Ω_h and are linear on each triangle $K_Q \in \mathcal{T}_h$. Let $\Pi_h : U \to U_h$ be the usual linear interpolator, and thus if $u \in W^{2,p}(\Omega)$,

$$|u - \Pi_h u|_{m,p} \leq C h^{2-m} |u|_{2,p}, \qquad m = 0, 1, \quad 1 \leq p \leq \infty, \tag{1.10}$$

where $|\cdot|_{m,p}$ is the usual seminorm of the Sobolev space $W^{m,p}(\Omega)$. This inequality can be obtained from its "polygonal" version using standard analysis [23] in the "skin layer" with the help of (1.7). Throughout the paper C will denote a generic constant independent of h and can have different values in different places. We use $||\cdot||_m$ and $|\cdot|_m$ for the norm $||\cdot||_{m,p}$ and the seminorm of $W^{m,p}(\Omega)$, respectively, when $p = 2$.

The test function space $V_h \subset L^2(\Omega)$ associated with the dual partition $\mathcal{T}_h^*$ is defined as the set of all piecewise constants. More specifically, let χ_{P_0} be the characteristic function of the set $K^*_{P_0}$ we have for $v_h \in V_h$

$$v_h = \sum_{P_0 \in \Omega_h^\circ} v_h(P_0) \chi_{P_0}. \tag{1.11}$$

Note that a test function is identically zero outside Ω_h. Define the transfer operator $\Pi_h^* : U_h \to V_h$ connecting the trial and test spaces as

$$\Pi_h^* w := \sum_{P_0 \in \Omega_h^\circ} w_h(P_0) \chi_{P_0}, \tag{1.12}$$

and hence

$$||w - \Pi_h^* w||_0 \leq C h |w|_1. \tag{1.13}$$

The approximate problem we consider is to find $u_h \in U_h$ such that

$$a^*(u_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h, \tag{1.14}$$

where

$$a^*(u_h, v_h) := \sum_{P_0 \in \Omega_h^\circ} v_h(P_0) a^*(u_h, \chi_{P_0}), \tag{1.15}$$

and

$$a^*(u_h, \chi_{P_0}) := -\int_{\partial K^*_{P_0}} (A \nabla u_h) \cdot \mathbf{n} ds + \int_{K^*_{P_0}} q u_h dx, \tag{1.16}$$

where $\mathbf{n}$ is an outward unit normal to $\partial K^*_{P_0}$, and $a^*(\cdot, \cdot)$ is bilinear by construction. Using the facts $n_1 ds = dx_2$ and $n_2 ds = -dx_1$ yields

$$\begin{aligned} a^*(u_h, \chi_{P_0}) &= -\int_{\partial K^*_{P_0}} \sum_{i,j=1}^{2} a_{ij} \frac{\partial u_h}{\partial x_j} n_i ds + \int_{K^*_{P_0}} q u_h dx \\ &= -\int_{\partial K^*_{P_0}} w_h^{(1)} dx_2 + \int_{\partial K^*_{P_0}} w_h^{(2)} dx_1 + \int_{K^*_{P_0}} q u_h dx, \end{aligned} \tag{1.17}$$

where n_i is the i-th component of the outward unit normal to ∂K_p^*, and

$$w_h^{(1)} := a_{11}\frac{\partial u_h}{\partial x_1} + a_{12}\frac{\partial u_h}{\partial x_2}, \tag{1.18}$$

$$w_h^{(2)} := a_{21}\frac{\partial u_h}{\partial x_1} + a_{22}\frac{\partial u_h}{\partial x_2}. \tag{1.19}$$

Let us relate our work to the existing literature. The basic idea of the finite volume method for general elliptic problems is to use the divergence theorem on the elliptic operator L of (1.1) to convert the double integral into a boundary integral as in (1.17). If one discretizes the boundary integral in (1.17) using finite differences, one gets the so-called finite volume difference methods [1, 22] or the generalized difference methods [15, 16, 17]. On the other hand if one uses finite element spaces in the discretization, one gets the so-called finite volume element methods [3, 4]. In both cases two grids dual to each other are used. More recently, Nicolaides [18] generalized the usual operators in vector analysis such as Div, Grad, and the Laplacian to Delaunay-Voronoi meshes. This class of methods is now termed the covolume method and has been successfully extended to practical fluid problems [13, 14, 19, 21]. See [20] for a survey of the covolume method. Porsching [25] initiated the so-called network method, which has also been extended to the Stokes problem [6, 12, 11] with rigorous analysis and to two fluid flow problems [24, 5]. In the network method the emphasis is to conserve mass or energy over control volumes. The meshes chosen do not have to be of the Delaunay-Voronoi type. In this paper we take barycenters in favor of circumcenters (the Delaunay-Voronoi mesh system uses circumcenters), since the maximum norm estimation is less amenable in the latter case. We shall refer to any finite volume method utilizing two grids as a covolume method since the last two methods mentioned above are now subsumed under the name the covolume method [20]. In all the covolume methods cited so far none has addressed maximum norm estimates for general elliptic or parabolic problems, which are crucial to studying their nonlinear counterpart where the coefficient matrix A becomes dependent on the solution. (However, some computational results in a discrete L^∞ norm were reported in [13, p. 160].) The approximation problem (1.14) has been considered by [16, 17] where convergence results in the H^1 and L^2 norms were demonstrated. However, we shall prove these results in a unified way. The main purpose of this paper is to provide convergence results in the maximum norm for (1.14) and for an accompanying approximate parabolic problem.

We now outline a central idea used in this paper to show convergence in L^2, H^1, and maximum norms. The idea, we think, is general enough to be useful for numerical analysts working in covolume methods. Our style of presenting it will follow that of the classical paper [23] on maxi-norm estimates in the finite element method. The central idea of analyzing the convergence of covolume methods is to reformulate (1.14) to find $u_h \in U_h$ such that

$$a^*(u_h, \Pi_h^* T_h) = (f, \Pi_h^* T_h) \qquad \forall T_h \in U_h,$$

which is a standard Galerkin method. With this association we can then tap into standard finite element analysis. A covolume method based on linear elements, if done properly, usually results in a system that is very close to the classical piecewise linear Galerkin method (more about this later). Comparison of the two systems then often leads to fruitful analysis. (This and similar ideas have been successfully exploited in [6, 12, 11, 8, 9, 10].) Now if one strives to carry out this program, one is very naturally led into considering the quantity (d for "deviation")

$$d(v - v_h, T_h) := a(v - v_h, T_h) - a^*(v - v_h, \Pi_h^* T_h), \tag{1.20}$$

where v is a "general" function, $v_h \in U_h, T_h \in U_h$. The basic observation is that (see (2.11))

$$d(v - v_h, T_h) = E_1 + E_2 + E_3 + E_4 + E_5, \tag{1.21}$$

where the E_i can be given various bounds that contain extra or "free" powers of h; something unexpected at first glance (at the E_i). Thus, for example, the bounds on various E_i take the following forms:

(B.E$_1$) $\quad C_A h||v - v_h||_1||T_h||_1,$

(B.E$_2$) $\quad C_A h[||v - v_h||_1 + h^{1/2}||v - v_h||_1^{1/2}||v||_2^{1/2}]||T_h||_1.$

Here C_A depends on $||\nabla A||_\infty$; it is 0 if the coefficient matrix A is constant.

(B.E$_3$) $\quad C_2 h^2||v||_{3,p}||T_h||_{1,p'}, \dfrac{1}{p} + \dfrac{1}{p'} = 1,$

(B.E$_4$) $\quad C_A h[||v - v_h||_1 + h^{1/2}||v - v_h||_1^{1/2}||v||_2^{1/2}]||T_h||_1,$

(B.E$_5$) $\quad C_q h||v - v_h||_0||T_h||_1.$

Here $C_q = 0$ if the function $q \equiv 0$.

Remark 1.1. See (2.12)-(2.26) for the derivation of these bounds.

To give a feel for the usefulness of this observation, let us take the case of

$$v \equiv 0, \quad A \text{ constant}, \quad q \equiv 0.$$

Then

$$d(u_h, T_h) = 0\,!$$

Thus the covolume approximation is given by

$$a(u_h, T_h) = (f, \Pi_h^* T_h) \qquad \forall T_h \in U_h,$$

whereas, for the ordinary Galerkin solution, $\tilde{u}_h$,

$$a(\tilde{u}_h, T_h) = (f, T_h) \qquad \forall T_h \in U_h.$$

Hence it is obvious that the covolume approximation can be viewd as a Galerkin method with a variational crime. In the general case,

$$a(u_h, T_h) + d(u_h, T_h) = (f, \Pi_h^* T_h)$$

with similar interpretation as two variational crimes. This view is very useful when dealing with the generalized Stokes problem (see [6, 12, 11] for more detail).

Now back to the issues of general estimates; take $v \equiv 0, v_h \in U_h$ and $T_h = v_h$ and apply (B.E$_1$), (B.E$_2$), (B.E$_4$), and (B.E$_5$) ((B.E$_3$) is void since $v \equiv 0$):

$$|d(v_h, v_h)| \le Ch||v_h||_1^2.$$

From this the coercivity (for h small enough) and boundedness of $a^*(\cdot, \Pi_h^* \cdot)$ follow (see Lemma 2.3).

Next, take $v \equiv 0, v_h = e_h := \tilde{u}_h - u_h$ ($\tilde{u}_h$ ordinary Galerkin) to find

$$\begin{aligned} ||e_h||_1^2 &\leq Ca^*(\tilde{u}_h - u_h, \Pi_h^* e_h) \\ &= C[a^*(\tilde{u}_h, \Pi_h^* e_h) - a^*(u_h, \Pi_h^* e_h)] \\ &= C[(f, e_h) - (f, \Pi_h^* e_h) - d(\tilde{u}_h, e_h)], \end{aligned}$$

and it follows immediately that

$$||e_h||_1^2 \leq Ch(||f||_0 + ||\tilde{u}_h||_1)||e_h||_1$$

so that, by the triangle inequality, $||u - u_h||_1 \leq Ch||f||_0$, which proves the H^1 convergence (see Lemma 2.5).

Similarly, we can derive L^2 convergence via a duality argument as follows. Note that

$$||e_h||_0 = \sup_{||\phi||_0=1} (e_h, \phi).$$

Given a ϕ with unit L^2–norm, let $L\psi = \phi, \psi = 0$ on $\partial\Omega$ and let $\tilde{\psi}_h$ be the Ritz projection of ψ. Thus

$$\begin{aligned} (e_h, \phi) &= a(e_h, \psi) = a(e_h, \tilde{\psi}_h) \\ &= a(u - u_h, \tilde{\psi}_h) \\ &= d(u - u_h, \tilde{\psi}_h) \\ &\leq C_A(h||u - u_h||_1 + h^{3/2}||u - u_h||_1^{1/2}||u||_2^{1/2}) \\ &\quad +C_2 h^2||u||_{3,p}||\tilde{\psi}_h||_{1,p'} \ (\tfrac{1}{p} + \tfrac{1}{p'} = 1) \\ &\quad +C_q h||u - u_h||_0. \end{aligned}$$

Here, $||\tilde{\psi}_h||_{1,p'} \leq C||\psi||_{1,p'} \leq C_p||\psi||_2$ (stability and Sobolev). Clearly, after some trivial manipulations, we obtain convergence in the L^2 norm.

The $W^{1,\infty}$ and L^∞ norm estimation follows the same vein but is more involved. The details can be found in Section 3. The organization of this paper is as follows. In Section 2 we list and prove preliminary lemmas and the H^1, L^2 norm convergence results. In Section 3 we derive maximum norm error estimates for the elliptic problems. The main results are contained in Theorem 3.1 (the max-norm error in the approximate solution is $O(h^2|\ln h|)$) and Theorem 3.2 (the max-norm error in the gradient is $O(h)$). The method of proof uses the above-mentioned central idea with the aid of the discrete Green's function. In Section 4 we give similar maximum norm estimates for parabolic equations.

2. Preliminaries

Define the discrete L^2 norm:

(2.1) $$||u_h||_{0,h} := ||\Pi_h^* u_h||_0 = \{\sum_{K_P^* \in \mathcal{T}_h^*} u_h^2(P) S_P^*\}^{1/2}.$$

Referring to Figure 2 and using the fact that Q are centers and M_i are midpoints, we have

(2.2) $$||u_h||_{0,h} = \{\frac{1}{3} \sum_{K_Q \in \mathcal{T}_h} [u_h^2(P_1) + u_h^2(P_2) + u_h^2(P_3)] S_Q\}^{1/2}.$$

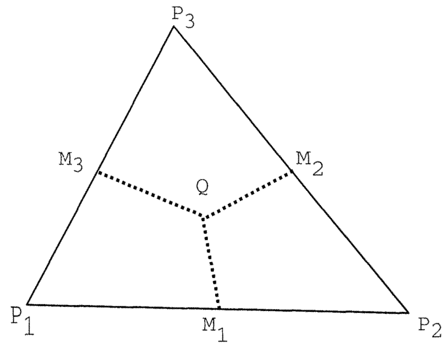

FIGURE 2. Primal triangular element with dual partition

Next define the discrete H^1 seminorm and norm:

$$|u_h|_{1,h} := \Big(\sum_{K_Q\in\mathcal{T}_h} |u_h|^2_{1,h,K_Q}\Big)^{1/2}, \tag{2.3}$$

$$|u_h|_{1,h,K} := \{[(\frac{\partial u_h}{\partial x_1}(Q))^2 + (\frac{\partial u_h}{\partial x_2}(Q))^2]S_Q\}^{1/2}, \tag{2.4}$$

$$||u||_{1,h} := \{||u_h||^2_{0,h} + |u|^2_{1,h}\}^{1/2}. \tag{2.5}$$

Lemma 2.1. *The two norms $|\cdot|_{1,h}$ and $|\cdot|_1$ are consistent, i.e., $|\cdot|_{1,h} = |\cdot|_1$, and $||\cdot||_{0,h}$ and $||\cdot||_{1,h}$ are equivalent to $||\cdot||_0$ and $||\cdot||_1$, respectively. Here the equivalence constants are independent of h.*

Proof. The first statement is easy to see since ∇u_h is constant over K_Q. As for the second statement, it suffices to show the equivalence of the L^2 norms. In reference to Figure 2, we have with $K = K_Q$

$$\begin{aligned}\int_K |u_h|^2 dx &= \frac{1}{3}[u_h^2(M_1) + u_h^2(M_2) + u_h^2(M_3)]S_Q\\ &= \frac{1}{12}[u_h^2(P_1) + u_h^2(P_2) + u_h^2(P_3) + (u_h(P_1) + u_h(P_2) + u_h(P_3))^2]S_Q.\end{aligned}$$

Summing over K yields

$$\frac{1}{4}||u_h||^2_{0,h} \le ||u_h||^2_0 \le ||u_h||^2_{0,h}.$$

□

Lemma 2.2. Π_h^* *is self-adjoint with respect to the L^2 inner product.*

$$(u_h, \Pi_h^*\bar{u}_h) = (\bar{u}_h, \Pi_h^* u_h), \quad \forall u_h, \bar{u}_h \in U_h. \tag{2.6}$$

Define

$$|||u_h|||_0 := (u_h, \Pi_h^* u_h)^{1/2}. \tag{2.7}$$

Then $|||\cdot|||_0$ and $||\cdot||_0$ are equivalent. Here the equivalence constants are independent of h.

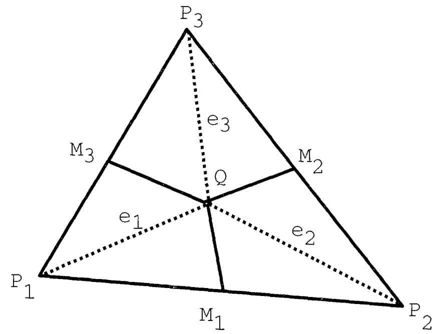

FIGURE 3. A triangular element K

Proof. In reference to Figure 3, for $i = 1, \dots, 3$, let e_i be the quadrilateral $P_iM_iQM_{i+2}$, ($M_5 = M_2, M_4 = M_1$) and λ_i be the Lagrange nodal basis functions associated with P_i, i.e., λ_1, λ_2, and λ_3 are the barycentric coordinates. Over a typical K write

$$u_h = \sum_{i=1}^{3} u_h(P_i)\lambda_i$$

(we will use local indices when there is no danger of confusion), and use (1.12) to obtain

$$\begin{aligned}
(u_h, \Pi_h^* \bar{u}_h) &= \sum_{K\in\mathcal{T}_h} \int_K u_h \Pi_h^* \bar{u}_h dx \\
&= \sum_{K\in\mathcal{T}_h} \sum_{l=1}^{3} \bar{u}_h(P_l) \int_{e_l} u_h dx \\
&= \sum_{K\in\mathcal{T}_h} \sum_{l=1}^{3} \bar{u}_h(P_l) \sum_{k=1}^{3} u_h(P_k) \int_{e_l} \lambda_k dx \\
&= \sum_{K\in\mathcal{T}_h} \sum_{k=1}^{3} \sum_{l=1}^{3} \bar{u}_h(P_l) u_h(P_k) \int_{e_k} \lambda_l dx \\
&= \sum_{K\in\mathcal{T}_h} \sum_{k=1}^{3} u_h(P_k) \sum_{l=1}^{3} \bar{u}_h(P_l) \int_{e_k} \lambda_l dx \\
&= (\bar{u}_h, \Pi_h^* u_h),
\end{aligned}$$

where we have interchanged the summations and used the fact that

$$\int_{e_k} \lambda_l dx = \int_{e_l} \lambda_k dx.$$

This last equality can be shown as follows. First it is easy to see that the triangle K is divided into six equal-area subtriangles. Use the three-vertices quadrature rule on linears to evaluate:

$$\begin{aligned}
\int_{e_1} \lambda_2 dx &= \int_{e_{1+}} \lambda_2 dx + \int_{e_{1-}} \lambda_2 dx \\
&= \frac{1}{3}(\lambda_2(P_1) + \lambda_2(Q) + \lambda_2(M_1))S_{e_{1+}} \\
&\quad + \frac{1}{3}(\lambda_2(P_1) + \lambda_2(Q) + \lambda_2(M_3))S_{e_{1-}} \\
&= \frac{1}{3}(0 + 1/3 + 1/2)S_{e_{1+}} + \frac{1}{3}(0 + 1/3 + 0)S_{e_{1-}},
\end{aligned}$$

where e_{1+} and e_{1-} are the two subtriangles that make up e_1. Since $S_{e_{1-}}, S_{e_{1+}}, S_{e_{2-}}$ and $S_{e_{2+}}$ are the same, we see that

$$\int_{e_1} \lambda_2 dx = \int_{e_2} \lambda_1 dx.$$

The other cases can be handled similarly since the underlying integrals only depend on the two areas as shown above. Finally, as a by-product, the equivalence of the two norms now follows by direct computation. □

Now let us derive the important relation (1.21) mentioned in Section 1. For $v \in H^2(\Omega), v_h, T_h \in U_h$, we have by Green's formula and the fact that T_h vanishes outside Ω_h that (see Figure 2)

$$\begin{aligned}
a(v - v_h, T_h) &= \sum_{K \in \mathcal{T}_h} \int_K \sum_{i,j=1}^{2} a_{ij} \frac{\partial(v - v_h)}{\partial x_j} \frac{\partial T_h}{\partial x_i} dx \\
&\quad + \int_{\Omega_h} q(v - v_h) T_h dx \\
&= \sum_K \int_K \sum_{i,j=1}^{2} [a_{ij}(x) - a_{ij}(Q)] \frac{\partial(v - v_h)}{\partial x_j} \frac{\partial T_h}{\partial x_i} dx \\
&\quad - \sum_K \int_K \sum_{i,j=1}^{2} a_{ij}(Q) \frac{\partial^2 v}{\partial x_i \partial x_j} T_h dx \\
&\quad + \sum_K \int_{\partial K} \sum_{i,j=1}^{2} a_{ij}(Q) \frac{\partial(v - v_h)}{\partial x_j} \cos\langle n, x_i \rangle T_h\, ds \\
&\quad + \int_{\Omega_h} q(v - v_h) T_h dx,
\end{aligned} \tag{2.8}$$

where Q is the center of K. Let K^v denote the set of all vertices of K. By Green's formula we have for $w \in H^2, w_h \in V_h$

$$\begin{aligned}
&\sum_{K\in \mathcal{T}_h} \int_K \frac{\partial^2 w}{\partial x_i \partial x_j} w_h dx \\
&= \sum_K \sum_{P\in K^v} \int_{K_P^*\cap K} \frac{\partial^2 w}{\partial x_i \partial x_j} w_h dx \\
&= \sum_K \sum_{P\in K^v} \left[-\int_{K_P^*\cap K} \frac{\partial w}{\partial x_j}\frac{\partial w_h}{\partial x_i} dx + \int_{\partial(K\cap K_P^*)} \frac{\partial w}{\partial x_j} \cos\langle n, x_i\rangle w_h ds \right] \\
&= \sum_K \sum_{P\in K^v} \int_{\partial(K\cap K_P^*)} \frac{\partial w}{\partial x_j} \cos\langle n, x_i\rangle w_h ds \\
&= \sum_K \sum_{P\in K^v} \left\{ \int_{\partial K_P^*\cap K} \frac{\partial w}{\partial x_j} \cos\langle n, x_i\rangle w_h ds + \int_{\partial K\cap K_P^*} \frac{\partial w}{\partial x_j} \cos\langle n, x_i\rangle w_h ds \right\}.
\end{aligned}$$

Hence, with $a_{ij}(Q)w$ in place of w in the above equation,

$$\begin{aligned}
&\sum_{K\in \mathcal{T}_h} \int_K a_{ij}(Q) \frac{\partial^2 w}{\partial x_i \partial x_j} w_h dx \\
&= \left\{ \sum_K \sum_{P\in K^v} \int_{\partial K_P^*\cap K} a_{ij}(Q) \frac{\partial w}{\partial x_j} \cos\langle n, x_i\rangle w_h ds \right\} \\
&\quad + \left\{ \sum_K \int_{\partial K} a_{ij}(Q) \frac{\partial w}{\partial x_j} \cos\langle n, x_i\rangle w_h ds. \right\}
\end{aligned} \tag{2.9}$$

Now argue as in deriving (2.8) and use (2.9) with $w = v - v_h$ and $w_h = \Pi_h^* T_h$ to obtain

$$\begin{aligned}
&a^*(v - v_h, \Pi_h^* T_h) \\
&= -\sum_K \sum_{P\in K^v} \int_{\partial K_P^*\cap K} \sum_{i,j=1}^{2} a_{ij} \frac{\partial(v - v_h)}{\partial x_j} \cos\langle n, x_i\rangle \Pi_h^* T_h ds \\
&\quad + \int_{\Omega_h} q(v - v_h)\Pi_h^* T_h dx \\
&= -\sum_K \sum_{P\in K^v} \int_{\partial K_P^*\cap K} \sum_{i,j=1}^{2} [a_{ij}(x) - a_{ij}(Q)] \frac{\partial(v - v_h)}{\partial x_j} \cos\langle n, x_i\rangle \Pi_h^* T_h ds \\
&\quad - \sum_K \int_K \sum_{i,j=1}^{2} a_{ij}(Q) \frac{\partial^2 v}{\partial x_i \partial x_j} \Pi_h^* T_h dx \\
&\quad + \sum_K \int_{\partial K} \sum_{i,j=1}^{2} a_{ij}(Q) \frac{\partial(v - v_h)}{\partial x_j} \cos\langle n, x\rangle \Pi_h^* T_h ds \\
&\quad + \int_{\Omega_h} q(v - v_h)\Pi_h^* T_h dx.
\end{aligned} \tag{2.10}$$

Hence

$$a(v-v_h,T_h)-a^*(v-v_h,\Pi_h^*T_h)=\sum_{i=1}^{5}E_i(v-v_h,T_h), \tag{2.11}$$

where

$$E_1(v-v_h,T_h)=\sum_K\int_K\sum_{i,j=1}^{2}[a_{ij}(x)-a_{ij}(Q)]\frac{\partial(v-v_h)}{\partial x_j}\frac{\partial T_h}{\partial x_i}dx, \tag{2.12}$$

$$\begin{aligned}E_2(v-v_h,T_h)=\sum_K\sum_{P\in K^v}\int_{\partial K_P^*\cap K}&\sum_{i,j=1}^{2}[a_{ij}(x)-a_{ij}(Q)]\\ &\cdot\frac{\partial(v-v_h)}{\partial x_j}\cos\langle n,x_i\rangle\Pi_h^*T_h\,ds,\end{aligned} \tag{2.13}$$

$$E_3(v-v_h,T_h)=-\sum_K\int_K\sum_{i,j=1}^{2}a_{ij}(Q)\frac{\partial^2 v}{\partial x_i\partial x_j}(T_h-\Pi_h^*T_h)dx, \tag{2.14}$$

$$\begin{aligned}&E_4(v-v_h,T_h)\\ &\quad=\sum_K\int_{\partial K}\sum_{i,j=1}^{2}a_{ij}(Q)\frac{\partial(v-v_h)}{\partial x_j}\cos\langle n,x_i\rangle(T_h-\Pi_h^*T_h)ds,\end{aligned} \tag{2.15}$$

$$E_5(v-v_h,T_h)=\int_{\Omega_h}q(v-v_h)(T_h-\Pi_h^*T_h)dx. \tag{2.16}$$

We are now in a position to show various bounds for E_i's introduced in the previous section. In view of the definition (2.12), bound (B.E$_1$) is straightforward since a_{ij} is in $W^{1,\infty}$. As for (B.E$_2$), from (2.13) $E_2(v-v_h,T_h)$ can be rewritten (see Figure 2)

$$\begin{aligned}E_2(v-v_h,T_h)=\sum_K\sum_{l=1}^{3}\int_{\overline{M_lQ}}&\sum_{i,j=1}^{2}[a_{ij}(x)-a_{ij}(Q)]\frac{\partial(v-v_h)}{\partial x_j}\\ &\times\cos\langle n,x_i\rangle ds\ [T_h(P_l)-T_h(P_{l+1})],\end{aligned} \tag{2.17}$$

where $P_4:=P_1$. The equality is obtained by noticing that each line segment M_lQ is traveled twice but in opposite orientations (once as $\overline{M_lQ}$, once as $\overline{QM_l}$) and then collecting the like-terms. By Taylor's expansion and the fact T_h is linear in K,

$$\begin{aligned}|T_h(P_l)-T_h(P_{l+1})|&=|\sum_{i=1}^{2}\frac{\partial T_h}{\partial x_i}[x_i(P_l)-x_i(P_{l+1})]|\\ &\le h(|\frac{\partial T_h}{\partial x_1}|+|\frac{\partial T_h}{\partial x_2}|)\le C|T_h|_{1,h,K}.\end{aligned} \tag{2.18}$$

On the other hand, by the Cauchy-Schwarz inequality

$$\int_{\overline{M_lQ}}|\frac{\partial(v-v_h)}{\partial x_i}|ds\le Ch^{1/2}\{\int_{\overline{M_lQ}}|\phi_i|^2ds\}^{1/2}, \tag{2.19}$$

where $\phi_i := \frac{\partial(v-v_h)}{\partial x_i}$. Use the trace theorem ([2, p. 37]) and a scaling argument to obtain

$$\begin{aligned}\int_{\overline{M_l Q}} |\phi_i|^2 ds &\le C(h^{-1}||\phi_i||_{0,K}^2 + ||\nabla\phi_i||_{0,K}||\phi_i||_{0,K}) \\ &\le C(h^{-1}|v-v_h|_{1,K}^2 + |v-v_h|_{2,K}|v-v_h|_{1,K}) \\ &= C(h^{-1}|v-v_h|_{1,K}^2 + |v|_{2,K}|v-v_h|_{1,K}).\end{aligned} \tag{2.20}$$

Collecting estimates, using Lemma 2.1 and the generalized Hölder's inequality, we have

$$|E_2(v-v_h, T_h)| \le Ch\{|v-v_h|_1|T_h|_1 + h^{1/2}|v-v_h|_1^{\frac{1}{2}}|v|_2^{\frac{1}{2}}|T_h|_1\}, \tag{2.21}$$

which implies (B.E$_2$).

Using proper quadratures for the two integrands and the fact that the quadrilaterals e_i of Figure 3 have equal area, it is easy to see that

$$\int_K (T_h - \Pi_h^* T_h)dx = 0 \qquad \forall T_h \in U_h.$$

Hence

$$|E_3| = |\sum_K \int_K \sum_{i,j=1}^2 a_{ij}(Q)[\frac{\partial^2 v(x)}{\partial x_i \partial x_j} - \mathcal{P}_K \frac{\partial^2 v}{\partial x_i \partial x_j}](T_h - \Pi_h^* T_h)dx|, \tag{2.22}$$

where $\mathcal{P}_K$ is the local L_2 projection to the space of piecewise constants. (Note that using $\mathcal{P}_K \frac{\partial^2 v}{\partial x_i \partial x_j}$ instead of $\frac{\partial^2 v}{\partial x_i \partial x_j}(Q)$ avoids asking v to be in C^2, as is done in some literature.) From this bound (B.E$_3$) follows easily.

As for the estimation of E_4, first note that $\frac{\partial v_h}{\partial x_j}\cos\langle n, x_i\rangle$ is constant along an edge L of the element K and that

$$\int_L (T_h - \Pi_h^* T_h)ds = 0. \tag{2.23}$$

Thus

$$\begin{aligned}&E_4(v-v_h, T_h) \\ &= \sum_K \int_{\partial K} \sum_{i,j=1}^2 a_{ij}(Q)\frac{\partial(v-v_h)}{\partial x_j}\cos\langle n, x_i\rangle(T_h - \Pi_h^* T_h)ds \\ &= \sum_K \int_{\partial K} \left[\sum_{i,j=1}^2 a_{ij}(Q)\frac{\partial v}{\partial x_j}\cos\langle n, x_i\rangle\right](T_h - \Pi_h^* T_h)ds.\end{aligned} \tag{2.24}$$

Let $\mathcal{E}$ be the collection of all the interior edges in the primal triangulation $\mathcal{T}_h$. (An interior edge does not lie on $\partial\Omega_h$.)

Using the boundary condition of T_h on $\partial\Omega_h$, continuity of $T_h - \Pi_h^* T_h$ and continuity of $\frac{\partial v}{\partial x_j}\cos\langle n, x_i\rangle$ across the edges in $\mathcal{E}$ (guaranteed by $v \in H^3(\Omega)$), we have

$$
\begin{aligned}
E_4 &= \sum_{L\in\mathcal{E}} \int_L \sum_{i,j=1}^{2} (a_{ij}(Q_L^+) - a_{ij}(Q_L^-)) \\
&\quad \times \frac{\partial v}{\partial x_j} \cos\langle n, x_i\rangle (T_h - \Pi_h^* T_h) ds \\
&= \sum_{L\in\mathcal{E}} \int_L \sum_{i,j=1}^{2} (a_{ij}(Q_L^+) - a_{ij}(Q_L^-)) \\
&\quad \times \left(\frac{\partial v}{\partial x_j} - \nu_j\right) \cos\langle n, x_i\rangle (T_h - \Pi_h^* T_h) ds,
\end{aligned} \tag{2.25}
$$

where Q_L^+ and Q_L^- are the centers of the two triangles sharing L as a common edge, and the addition of a constant ν_j is due to (2.23). Now we choose ν_j as

$$\nu_j := \frac{1}{2}\left(\frac{\partial v_h^+}{\partial x_j} + \frac{\partial v_h^-}{\partial x_j}\right),$$

where v_h^+ (resp. v_h^-) is the restriction of v_h to the left (resp. right) triangle K_L (resp. K_R).

Observe that

$$\sum_{L\in\mathcal{E}} \int_L \sum_{i,j=1}^{2} (a_{ij}(Q_L^+) - a_{ij}(Q_L^-)) \left(\frac{\partial v}{\partial x_j} - \frac{\partial v_h^\sigma}{\partial x_j}\right) \cos\langle n, x_i\rangle (T_h - \Pi_h^* T_h) ds,$$

where $\sigma = +$ or $-$ resembles E_2. The technique used in deriving (2.20) yields

$$\left(\int_L (T_h - \Pi_h^* T_h)^2 ds\right)^{1/2} \le Ch^{1/2} ||T_h||_{1,K}.$$

Thus as in deriving out (2.21), we have bound (B.E$_4$)

$$|E_4(v - v_h, T_h)| \le Ch\{|v - v_h|_1 ||T_h||_1 + h^{1/2} |v - v_h|_1^{\frac{1}{2}} |v|_2^{\frac{1}{2}} ||T_h||_1\}. \tag{2.26}$$

Finally, bound (B.E$_5$) follows from (2.16) easily. The following lemma is now proved in view of the central observation in Section 1.

Lemma 2.3. *There exist positive constants h_0, α, M such that for $0 \le h \le h_0$*

$$a^*(u_h, \Pi_h^* u_h) \ge \alpha ||u_h||_1^2, \quad \forall u_h \in U_h, \tag{2.27}$$

$$|a^*(u_h, \Pi_h^* T_h)| \le M ||u_h||_1 ||T_h||_1, \quad \forall u_h, T_h \in U_h. \tag{2.28}$$

For covolume methods we seldom have a symmetric bilinear form $a^*(\cdot, \Pi_h^* \cdot)$ even though $a(\cdot,\cdot)$ is. However, we have a lemma which measures how far the bilinear form $a^*(\cdot, \Pi_h^* \cdot)$ is from being symmetric. This lemma will be used in the parabolic problem.

Lemma 2.4. *There exist positive constants M, h_0 such that for $0 < h \le h_0$*

$$|a^*(u_h, \Pi_h^* T_h) - a^*(T_h, \Pi_h^* u_h)| \le Mh ||u_h||_1 ||T_h||_1 \quad \forall u_h, T_h \in U_h. \tag{2.29}$$

Proof. Use (1.20) and the triangle inequality to derive

$$|a^*(u_h, \Pi_h^* T_h) - a^*(T_h, \Pi_h^* u_h)| \le |d(u_h, T_h) - d(T_h, u_h)|.$$

Invoking proper bounds for $d(\cdot,\cdot)$ completes the proof. □

The next lemma is proved in Section 1.

Lemma 2.5. *The solution of u_h of the problem* (1.14) *and the exact solution u of* (1.1) *satisfy*

$$\begin{aligned} ||u-u_h||_1 &\le Ch||u||_2, && (2.30)\\ ||u-u_h||_0 &\le Ch^2||u||_{3,p} \qquad (p>1), && (2.31)\end{aligned}$$

whenever the right-hand sides make sense.

Given any $z \in \bar{\Omega}$, we define $G_z^h \in U_h$ to be the discrete Green's function associated with the form $a(\cdot,\cdot)$ if

$$a(G_z^h, w_h) = w_h(z) \quad \forall w_h \in U_h. \tag{2.32}$$

Lemma 2.6. *The function G_z^h possesses the following properties* [26, 27]:

$$||G_z^h||_1 \le C|\ln h|^{1/2}. \tag{2.33}$$

Let v be a given unit vector (direction) and let Δz be any vector parallel to v. Then we define

$$\partial_z G_z^h := \lim_{\Delta z \to 0} \frac{G_{z+\Delta z}^h - G_z^h}{|\Delta z|}. \tag{2.34}$$

Lemma 2.7. *The derivative $\partial_z G_z^h \in U_h$ has the following properties* [27]:

$$\begin{aligned} a(\partial_z G_z^h, v_h) &= \partial_z v_h(z) \quad \forall v_h \in U_h, && (2.35)\\ ||\partial_z G_z^h||_1 &\le Ch^{-1}. && (2.36)\end{aligned}$$

Lemma 2.8. *Let u and u_h be the solutions of* (1.1) *and* (1.14), *respectively. Then*

$$a^*(u-u_h, v_h) = 0, \quad \forall v_h \in V_h. \tag{2.37}$$

3. Maximum norm estimates for an elliptic problem

Theorem 3.1. *Let u be the solution of* (1.1) *and u_h be the solution of* (1.14). *Then*

$$||u-u_h||_{0,\infty} \le Ch^2|\ln h| \; [||u||_3 + ||u||_{2,\infty}], \tag{3.1}$$

provided that $u \in H_0^1(\Omega) \cap W^{2,\infty}(\Omega) \cap H^3(\Omega)$.

Proof. Let $\tilde{u}_h$ be the ordinary Galerkin of (1.1).

$$||u-u_h||_{0,\infty} \le ||u-\tilde{u}_h||_{0,\infty} + ||\tilde{u}_h - u_h||_{0,\infty}. \tag{3.2}$$

Since it is well known [26] that the maximum norm error in $\tilde{u}_h$ is bounded by $Ch^2|\ln h|\,||u||_{2,\infty}$, it suffices to estimate $e_h := \tilde{u}_h - u_h$. By the definition of the discrete Green's function and (2.37)

$$\begin{aligned} e_h(z) &= a(e_h, G_z^h)\\ &= a(u-u_h, G_z^h)\\ &= d(u-u_h, G_z^h).\end{aligned}$$

Now we estimate $E_i(u-u_h, G_z^h), i = 1, \ldots, 5$. By (B.E$_1$), (2.33) and Lemma 2.5,

$$|E_1(u-u_h, G_z^h)| \le Ch||u-u_h||_1||G_z^h||_1 \le Ch^2|\ln h|^{1/2}||u||_2. \tag{3.3}$$

By (B.E$_2$), (2.33) and Lemma 2.5,

$$\begin{aligned} |E_2(u-u_h,G_z^h)| &\le C_A h[||u-u_h||_1 + h^{1/2}||u-u_h||_1^{1/2}||u||_2^{1/2}]||G_z^h||_1 \\ &\le Ch^2|\ln h|^{\frac{1}{2}}||u||_2. \end{aligned} \tag{3.4}$$

By (B.E$_3$) and (2.33),

$$\begin{aligned} |E_3(u-u_h,G_z^h)| &\le Ch^2||u||_3||G_z^h||_1 \\ &\le Ch^2|\ln h|^{\frac{1}{2}}||u||_3, \end{aligned} \tag{3.5}$$

$$|E_4(u-u_h,G_z^h)| \le Ch^2|\ln h|^{\frac{1}{2}}||u||_2.$$

Finally

$$\begin{aligned} |E_5(u-u_h,G_z^h)| &\le Ch||u-u_h||_0||G_z^h||_1 \\ &\le Ch^2||u||_3. \end{aligned} \tag{3.6}$$

□

Theorem 3.2. *Under the hypotheses of Theorem 3.1*

$$||u-u_h||_{1,\infty} \le Ch\left[||u||_3 + ||u||_{2,\infty}\right]. \tag{3.7}$$

Proof. The proof parallels the development in Theorem 3.1. Since it is well known [26] that the error in $\tilde{u}_h$ is bounded by $Ch||u||_{2,\infty}$, it suffice to estimate $e_h := \tilde{u}_h - u_h$ in $W^{1,\infty}$. As before

$$\begin{aligned} \partial_z e_h(z) &= a(e_h, \partial_z G_z^h) \\ &= a(u-u_h, \partial_z G_z^h) \\ &= d(u-u_h, \partial_z G_z^h) \\ &= \sum_{i=1}^{5} E_i(u-u_h, \partial_z G_z^h), \end{aligned}$$

where

$$\begin{aligned} |E_1(u-u_h,\partial_z G_z^h)| &\le Ch||u-u_h||_1||\partial_z G_z^h||_1 \\ &\le Ch||u||_2, \\ |E_2(u-u_h,\partial_z G_z^h)| &\le Ch^2||u||_2||\partial_z G_z^h||_1 \\ &\le Ch||u||_2, \\ |E_3(u-u_h,\partial_z G_z^h)| &\le Ch^2||u||_3||\partial_z G_z^h||_1 \\ &\le Ch||u||_3, \\ |E_4(u-u_h,\partial_z G_z^h)| &\le Ch||u||_2, \\ |E_5(u-u_h,\partial_z G_z^h)| &\le C||u-u_h||_0||\partial_z G_z^h||_1 \\ &\le Ch||u||_3. \end{aligned} \tag{3.8}$$

Combining all the above inequalities completes the proof. □

4. Maximum norm estimates for parabolic problems

Consider the associated parabolic problem to (1.1)-(1.2):

$$u_t + Lu = f(x,t), \quad (x,t) \in \Omega \times (0,T] \tag{4.1}$$

$$u = 0, \quad (x,t) \in \partial\Omega \times (0,T] \tag{4.2}$$

$$u = u_0(x), \quad t = 0, x \in \Omega, \tag{4.3}$$

where L is the elliptic operator of (1.1) and $u_t := \frac{\partial u}{\partial t}$. The domain Ω has the primal partition $\mathcal{T}_h$ and dual partition $\mathcal{T}_h^*$ of the types specified in Section 1. The trial and test spaces are still $U_h \subset H_0^1(\Omega)$ and $V_h \subset L^2(\Omega)$, respectively. Consider the time-continuous approximation to (4.1)-(4.3):
Find $u_h := u_h(\cdot, t) \in U_h, 0 \le t \le T$ such that

$$(u_{h,t}, v_h) + a^*(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h, t > 0, \tag{4.4}$$

$$u_h(x,0) = u_{0h}(x), \quad x \in \Omega, \tag{4.5}$$

where the approximate initial condition u_{0h} is the elliptic projection (see (4.8)) of the exact initial function to be specified in (4.15).

Theorem 4.1. *Let u and u_h be the solutions of* (4.1)-(4.3) *and* (4.4)-(4.5), *respectively. Then for $p > 1$*

$$||u - u_h||_{L^\infty(L^\infty)} \le Ch^2 |\ln h| \{||u||_{L^\infty(H^3)} + ||u||_{L^\infty(W^{2,\infty})} + ||u_t||_{L^2(W^{3,p})}\}, \tag{4.6}$$

$$||u - u_h||_{L^\infty(W^{1,\infty})} \le Ch\{||u||_{L^\infty(H^3)} + ||u||_{L^\infty(W^{2,\infty})} + ||u_t||_{L^2(W^{3,p})}\}, \tag{4.7}$$

where $L^\infty(L^\infty) := L^\infty(0,T; L^\infty(\Omega)), L^\infty(H^3) := L^\infty(0,T; H^3(\Omega))$.

Proof. Introduce the self-adjoint operator $R_h : H^2(\Omega) \cap H_0^1(\Omega) \to U_h$ defined by

$$a^*(R_h u, v_h) = a^*(u, v_h) \quad \forall v_h \in V_h. \tag{4.8}$$

By Lemma 2.5, and Theorems 3.1 and 3.2,

$$||(u - R_h u)_t||_0 \le Ch^2 ||u_t||_{3,p}, \quad p > 1, \tag{4.9}$$

$$||(u - R_h u)||_{0,\infty} \le Ch^2 |\ln h| \left[||u||_3 + ||u||_{2,\infty}\right], \tag{4.10}$$

$$||(u - R_h u)||_{1,\infty} \le Ch \left[||u||_3 + ||u||_{2,\infty}\right]. \tag{4.11}$$

Write $u - u_h = (u - R_h u) + (R_h u - u_h) := \eta + \xi$. It suffices to estimate ξ. By (4.1)-(4.4) and (4.8),

$$(\xi_t, v_h) + a^*(\xi, v_h) = -(\eta_t, v_h), \forall v_h \in V_h. \tag{4.12}$$

Set $v_h = \Pi_h^* \xi_t$ and use (2.7) to obtain

$$\begin{aligned} |||\xi_t|||_0^2 &+ \frac{1}{2}\frac{d}{dt} a^*(\xi, \Pi_h^* \xi) \\ &= -(\eta_t, \Pi_h^* \xi_t) + \frac{1}{2}[a^*(\xi_t, \Pi_h^* \xi) - a^*(\xi, \Pi_h^* \xi_t)]. \end{aligned} \tag{4.13}$$

By Lemma 2.4, an inverse inequality, and Lemma 2.2,

$$\begin{aligned} |a^*(\xi_t, \Pi_h^* \xi) - a^*(\xi, \Pi_h^* \xi_t)| &\le Ch ||\xi_t||_1 ||\xi||_1 \\ &\le C||\xi_t||_0 ||\xi||_1 \le |||\xi_t|||_0^2 + C||\xi||_1^2, \end{aligned}$$

where we have used the ϵ-inequality $ab \leq \epsilon a^2 + \frac{1}{4\epsilon}b^2$ for positive ϵ, a, b. Taking ϵ small enough to absorb the ξ_t term on the right-hand side into the left-hand side, we have

$$\frac{d}{dt}a^*(\xi, \Pi_h^*\xi) \leq C(||\eta_t||_0^2 + ||\xi||_1^2). \tag{4.14}$$

Set

$$u_{0h} = R_h u(0) \tag{4.15}$$

so that $\xi(0) = 0$. Integrate (4.14) and use Lemma 2.3 to get

$$\alpha||\xi||_1^2 \leq a^*(\xi, \Pi_h^*\xi) \leq C\int_0^t (||\eta_t||_0^2 + ||\xi||_1^2)d\tau. \tag{4.16}$$

Use (4.9) and the Gronwall's inequality to get

$$||\xi||_1 \leq Ch^2||u_t||_{L^2(W^{3,p})}, \quad p > 1. \tag{4.17}$$

From the asymptotic Sobolev inequality ([23, p. 274]), we have

$$||\xi||_{0.\infty} \leq C|\ln h|^{\frac{1}{2}}||\nabla\xi||_0 \leq Ch^2|\ln h|^{\frac{1}{2}}||u_t||_{L^2(W^{3,p})}. \tag{4.18}$$

Combine (4.10) and (4.18) to get (4.6) and then use an inverse inequality to get

$$||\xi||_{1,\infty} \leq Ch^{-1}||\xi||_1 \leq Ch||u_t||_{L^2(W^{3,p})}. \tag{4.19}$$

Noting (4.11) derives (4.7) completes the proof. □

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